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Summability of Power Series by Non-regular Nörlund-Methods

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Given a regular Nörlund-method (N, p) one can prove that the sequence $\{\sigma_n(z)\}$ of the Nörlund-transforms of a power series $f(z) = \sum_{v=0}^{\infty} a_v z^v$ with radius of convergence $r = 1$ converges in at most countably many points outside the unit disc. In this paper we show that for a class of non-regular Nörlund-methods the sequence $\{\sigma_n(z)\}$ converges to an analytic function in a disc which strictly contains the unit disc, and the convergence is uniform on any compact subset of this disc.

1. INTRODUCTION

As is well known, a Nörlund-method is defined in the following way.

DEFINITION 1. Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of complex numbers such that $P_n := \sum_{v=0}^{n} p_v \neq 0$ for all $n \in \mathbb{N}_0$. Then the triangular matrix $A = (\alpha_{nv})$ with the elements

$$\alpha_{nv} = \frac{p_{n-v}}{P_n} \quad \text{for } 0 \leqslant v \leqslant n \qquad \text{and} \qquad \alpha_{nv} = 0 \quad \text{for } v > n$$

generates a summability method which is called Nörlund-method (N, p).

Many different items about Nörlund-methods have been discussed in the literature (see, for example, [1, 2, 4–7, 11]). The present paper has a connection to a theorem due to Léja [8] and results of Luh [9]. Léja proved that for a regular Nörlund-method the sequence $\{\sigma_n(z)\}$ of the Nörlund-transforms of power series $f(z) = \sum_{v=0}^{\infty} a_v z^v$, with radius of convergence 1, converges in at most countably many points outside the unit disc $\mathbb{D} = \{z : |z| < 1\}$. And it was shown by Luh that for a regular (N, p)-method we have $\overline{\lim}_{n\to\infty} \{\max_{\Gamma} |\sigma_n(z)|\}^{1/n} = R$ for each closed arc Γ on $|z| = R > 1$.

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Neglecting the proposition of regularity for the Nörlund-method, we will show that the sequence $\{\sigma_n(z)\}$ can converge compactly (i.e., uniformly on each compact subset) to an analytic function in a disc which strictly contains the unit disc. Furthermore we will investigate the growth of $\{\sigma_n(z)\}$ outside the domain of convergence, which will lead us to results analogous to those of Léja and Luh.

2. Notations and Some Properties of (N, p)-Methods

Let $f(z)=\sum_{\nu=0}^{\infty} a_\nu z^\nu$ be a power series with radius of convergence 1 and let us consider its Nörlund-transforms

$$\sigma_n(z) := \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} s_\nu(z), \qquad \text{where} \quad s_\nu(z) = \sum_{\mu=0}^{\nu} a_\mu z^\mu.$$

We say that the power series $f(z)=\sum_{\nu=0}^{\infty} a_\nu z^\nu$ is summable by the method (N, p) in a point $z_0 \in \mathbb{C}$ if the sequence $\{\sigma_n(z_0)\}$ converges, and we say that the power series is compactly summable by the method (N, p) in a domain G to a function $\sigma(z)$ if $\{\sigma_n(z)\}$ converges uniformly to $\sigma(z)$ on each compact subset of G (notation $\sigma_n(z) \underset{G}{\Longrightarrow} \sigma(z)$).

We first give some results about special properties of Nörlund-methods.

From the theorem of Toeplitz, Schur, and Silverman (see, e.g., [10, p. 11]) it follows immediately that a Nörlund-method (N, p) is regular if and only if the following two conditions hold:

$$\lim_{n\to\infty} \frac{p_n}{P_n} = 0 \qquad \text{and} \qquad \sup_n \frac{1}{|P_n|} \sum_{\nu=0}^{n} |p_\nu| < \infty.$$

From the following theorem we can deduce that for each $\alpha \in \mathbb{C}$ there exists a sequence $\{p_n\}$ such that $\lim_{n\to\infty} p_n/P_n = \alpha$.

Theorem 1. *Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence of complex numbers with $\alpha_0 = 1$ $\alpha_n \neq 1$ for all $n \in \mathbb{N}$; then there exists a sequence $\{p_n\}_{n=0}^{\infty}$, $p_n \in \mathbb{C}$, with the properties*

$$P_n = \sum_{\nu=0}^{n} p_\nu \neq 0 \qquad \textit{and} \qquad \frac{p_n}{P_n} = \alpha_n.$$

We omit the simple proof.

For a Nörlund-method (N, p) which is defined by a sequence $p=\{p_n\}_{n=0}^{\infty}$ we now show that the convergence of the sequence $\{p_n/P_n\}_{n=0}^{\infty}$ implies the convergence of $\{p_{n-\nu}/P_n\}_{n=0}^{\infty}$ for each fixed $\nu \in \mathbb{N}_0$.

THEOREM 2. *Let (N, p) be a Nörlund-method with* $\lim_{n\to\infty} p_n/P_n = \alpha$ $(\alpha \in \mathbb{C})$. *Then for each fixed* $\nu \in \mathbb{N}_0$ *we have*

$$\lim_{n\to\infty} \frac{p_{n-\nu}}{P_n} = \alpha(1-\alpha)^\nu.$$

Proof. For $\nu = 0$ there is nothing left to prove. Now let $\nu \in \mathbb{N}$ be fixed; then for $n \geqslant \nu$ we get

$$\frac{p_{n-\nu}}{P_n} = \frac{p_{n-\nu}}{P_{n-\nu}} \cdot \prod_{k=0}^{\nu-1} \frac{P_{n-k-1}}{P_{n-k}} = \frac{p_{n-\nu}}{P_{n-\nu}} \cdot \prod_{k=0}^{\nu-1} \left(1 - \frac{p_{n-k}}{P_{n-k}}\right)$$

from which together with $\lim_{n\to\infty} p_n/P_n = \alpha$ the statement of our theorem follows.

Our next result shows that there is a connection between the convergence of $\{p_n/P_n\}$ and the sequence $\{|P_n|^{1/n}\}$.

THEOREM 3. *If* $\lim_{n\to\infty} p_n/P_n = \alpha$ $(\alpha \in \mathbb{C})$ *then we have* $\lim_{n\to\infty} |P_n|^{1/n} = 1/|1-\alpha|$.

Proof. From $\lim_{n\to\infty} p_n/P_n = \alpha$ and $p_n/P_n = 1 - P_{n-1}/P_n$ we get $\lim_{n\to\infty} P_{n-1}/P_n = 1 - \alpha$ which implies our assertion.

3. (N, p)-SUMMABILITY OF THE GEOMETRIC SEQUENCE AND POWER SERIES

It has been shown in [3] that the behaviour of matrix-transforms of $\{z^n\}$ is of great relevance for the behaviour of the considered matrix-transforms of power series. We therefore first examine the (N, p)-transforms of the geometric sequence $\{z^n\}$ which are given by

$$\tau_n(z) := \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} z^\nu \qquad (n \in \mathbb{N}_0).$$

We shall prove the following result.

THEOREM 4. *Let (N, p) be a Nörlund-method and $\alpha \in \mathbb{C}$. Then the following two statements are equivalent:*

(i) $\lim_{n\to\infty} p_n/P_n = \alpha$.

(ii) *The (N, p)-transforms of the geometric sequence $\{\tau_n(z)\}$ are compactly convergent in $|z| < 1/|1-\alpha|$ to the limit function* $\tau(z) = \alpha/(1-(1-\alpha)z)$.

Proof. 1. We assume that (N, p) is a Nörlund-method which satisfies

$$\tau_n(z) = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} z^\nu \underset{|z|<1/|1-\alpha|}{\Longrightarrow} \tau(z) = \frac{\alpha}{1-(1-\alpha)z}.$$

(a) For $\alpha \neq 1$ the limit function has the Taylor expansion $\tau(z) = \alpha \sum_{\nu=0}^{\infty} (1-\alpha)^\nu z^\nu$ in $|z| < 1/|1-\alpha|$. Since $\{\tau_n(z)\}$ converges compactly to $\tau(z)$ in $|z| < 1/|1-\alpha|$, we obtain

$$\tau_n(0) = \frac{p_n}{P_n} \to \tau(0) = \alpha \qquad (n \to \infty).$$

(b) For $\alpha = 1$ our assumption is that $\tau_n(z) \underset{\mathbb{C}}{\Longrightarrow} 1$, from which we easily conclude that $\lim_{n\to\infty} p_n/P_n = 1$.

2. We assume that (N, p) is a Nörlund-method which satisfies $\lim_{n\to\infty} p_n/P_n = \alpha$.

(a) For $\alpha \neq 1$ we show that $\{\tau_n(z)\}$ converges compactly to $\tau(z) = \alpha/(1-(1-\alpha)z) = \sum_{\nu=0}^{\infty} \alpha(1-\alpha)^\nu z^\nu$ in $|z| < 1/|1-\alpha|$.

(i) Given r with $0 < r < 1/|1-\alpha|$ we choose $\varepsilon > 0$ such that $(|1-\alpha|+\varepsilon)r < 1$. Then there exists an $N \in \mathbb{N}$ such that $\sum_{\nu=N+1}^{\infty} (|1-\alpha|\, r)^\nu < \varepsilon/|\alpha|$ and $\sum_{\nu=N+1}^{\infty} \{|1-\alpha|+\varepsilon)r\}^\nu < \varepsilon/(|\alpha|+\varepsilon)$. For $n > N$ we obtain

$$\begin{aligned}
\max_{|z|\leqslant r} |\tau_n(z) - \tau(z)| &\leqslant \sum_{\nu=0}^{n} \left| \frac{p_{n-\nu}}{P_n} - \alpha(1-\alpha)^\nu \right| r^\nu + |\alpha| \sum_{\nu=n+1}^{\infty} (|1-\alpha|\, r)^\nu \\
&\leqslant \sum_{\nu=0}^{N} \left| \frac{p_{n-\nu}}{P_n} - \alpha(1-\alpha)^\nu \right| r^\nu \\
&\quad + \sum_{\nu=N+1}^{n} \left| \frac{p_{n-\nu}}{P_n} \right| r^\nu + \sum_{\nu=N+1}^{n} |\alpha|\, (|1-\alpha|\, r)^\nu + \varepsilon \\
&\leqslant \sum_{\nu=0}^{N} \left| \frac{p_{n-\nu}}{P_n} - \alpha(1-\alpha)^\nu \right| r^\nu + \sum_{\nu=N+1}^{n} \left| \frac{p_{n-\nu}}{P_n} \right| r^\nu + 2\varepsilon.
\end{aligned}$$

By Theorem 2 the first sum converges to zero for $n \to \infty$.

(ii) To calculate the term $\sum_{\nu=N+1}^{n} |p_{n-\nu}/P_n|\, r^\nu$ we first look at the convergence of the sequence $\{p_{n-\nu}/P_n\}_{n=0}^{\infty}$ (ν fixed). By assumption we have $\lim_{n\to\infty} p_n/P_n = \alpha$ and therefore to $\varepsilon > 0$ given above there exists an $N_1 \in \mathbb{N}$ such that for all $n > N_1$ we have $|p_n/P_n - \alpha| < \varepsilon$. For $n > N_1$ and each ν with $0 \leqslant \nu \leqslant n - N_1$ we get

$$\left|\frac{p_{n-\nu}}{P_n}\right| = \left|\frac{p_{n-\nu}}{P_{n-\nu}}\right| \cdot \prod_{k=0}^{\nu-1} \left|\frac{P_{n-k-1}}{P_{n-k}}\right|$$
$$= \left|\frac{p_{n-\nu}}{P_{n-\nu}}\right| \cdot \prod_{k=0}^{\nu-1} \left|1 - \frac{p_{n-k}}{P_{n-k}}\right|$$
$$\leqslant (|\alpha| + \varepsilon)(|1-\alpha| + \varepsilon)^{\nu}.$$

If we choose N_1 large enough, then by Theorem 3 we obtain for $n > N_1$

$$|P_n| > \frac{1}{(|1-\alpha| + \varepsilon)^n}.$$

(iii) For all $n > N + N_1$ we now have

$$\sum_{\nu=N+1}^{n} \left|\frac{p_{n-\nu}}{P_n}\right| r^{\nu} = \sum_{\nu=N+1}^{n-N_1-1} \left|\frac{p_{n-\nu}}{P_n}\right| r^{\nu} + \sum_{\nu=n-N_1}^{n} \left|\frac{p_{n-\nu}}{P_n}\right| r^{\nu}$$
$$\leqslant (|\alpha| + \varepsilon) \sum_{\nu=N+1}^{n-N_1-1} (|1-\alpha| + \varepsilon)^{\nu} r^{\nu} + \frac{1}{|P_n|} \cdot \sum_{\mu=0}^{N_1} |p_{\mu}| r^{n-\mu}$$
$$\leqslant (|\alpha| + \varepsilon) \sum_{\nu=N+1}^{\infty} \{(|1-\alpha| + \varepsilon) r\}^{\nu}$$
$$+ \{(|1-\alpha| + \varepsilon) r\}^{n} \cdot \sum_{\mu=0}^{N_1} \frac{|p_{\mu}|}{r^{\mu}}$$
$$\leqslant \varepsilon + \{(|1-\alpha| + \varepsilon) r\}^{n} \cdot M,$$

where $M = \sum_{\mu=0}^{N_1} |p_{\mu}/r^{\mu}|$ is a constant. Since $(|1-\alpha| + \varepsilon) r < 1$ it follows that $\sum_{\nu=N+1}^{n} |p_{n-\nu}/P_n| \, r^{\nu}$ tends to zero for $n \to \infty$. Hence, together with (i) we obtain that

$$\max_{|z| \leqslant r} |\tau_n(z) - \tau(z)| \to 0 \qquad (n \to \infty),$$

and because r, $0 < r < 1/|1-\alpha|$, was arbitrary we have

$$\tau_n(z) \xrightarrow[|z| < 1/|1-\alpha|]{} \tau(z).$$

(b) Let $\alpha = 1$. We now show that $\{\tau_n(z)\}$ is compactly convergent to $\tau(z) \equiv 1$ in $\mathbb{C}$. Let $r \in (0, \infty)$ be arbitrary but fixed and $\varepsilon > 0$ such that $r \cdot \varepsilon < 1$. Like in (ii) of part (a) we can find an $N_2 \in \mathbb{N}$ such that $|p_{n-\nu}/P_n| < (1+\varepsilon) \cdot \varepsilon^{\nu}$ for all $n > N_2$ and $0 \leqslant \nu \leqslant n - N_2$ and furthermore by Theorem 3 that $|P_n| > K^n$, where $K > 1$ is a constant with $r/K < 1$. For $n > N_2$ we obtain

$$\begin{aligned}
\max_{|z|\leqslant r} |\tau_n(z)-1| &= \max_{|z|\leqslant r} \left| \sum_{v=0}^{n} \frac{p_{n-v}}{P_n} z^v - 1 \right| \\
&\leqslant \left| \frac{p_n}{P_n} - 1 \right| + \sum_{v=1}^{n} \left| \frac{p_{n-v}}{P_n} \right| r^v \\
&= \left| \frac{p_n}{P_n} - 1 \right| + \sum_{v=1}^{n-N_2-1} \left| \frac{p_{n-v}}{P_n} \right| r^v + \sum_{v=n-N_2}^{n} \left| \frac{p_{n-v}}{P_n} \right| r^v \\
&\leqslant \left| \frac{p_n}{P_n} - 1 \right| + (1+\varepsilon) \cdot \sum_{v=1}^{n-N_2-1} (\varepsilon r)^v + \frac{r^n}{|P_n|} \sum_{\mu=0}^{N_2} \frac{|p_\mu|}{r^\mu} \\
&\leqslant \left| \frac{p_n}{P_n} - 1 \right| + (1+\varepsilon) \cdot \left\{ \frac{1}{1-\varepsilon r} - 1 \right\} + \left(\frac{r}{K} \right)^n \cdot \sum_{\mu=0}^{N_2} \frac{|p_\mu|}{r^\mu}.
\end{aligned}$$

As n tends to infinity we have $|p_n/P_n - 1| \to 0$ and $(r/K)^n \to 0$ and because $\varepsilon > 0$ was arbitrarily small we have

$$\max_{|z|\leqslant r} |\tau_n(z) - 1| \to 0 \qquad (n \to \infty)$$

from which it follows that

$$\tau_n(z) \underset{\mathbb{C}}{\Longrightarrow} 1.$$

This proves the theorem.

From this result we can deduce the following theorem about the Nörlund-summability of power series.

THEOREM 5. *Let be $\alpha \in \mathbb{C}$ and (N, p) a Nörlund-method. Then the following two statements are equivalent*:

(i) $\lim_{n\to\infty} p_n/P_n = \alpha$.

(ii) *Each power series $f(z) = \sum_{v=0}^{\infty} a_v z^v$ with radius of convergence* 1 *is compactly summable (N, p) in $|z| < 1/|1-\alpha|$.*

For a proof see [3, pp. 38–45].
We will give some remarks and an example related to our last theorem.

Remarks. 1. If (N, p) is a Nörlund-method with $\lim_{n\to\infty} p_n/P_n = \alpha$ then by Theorem 5 each power series $f(z) = \sum_{v=0}^{\infty} a_v z^v$ with radius of convergence 1 is compactly summable (N, p) in $|z| < 1/|1-\alpha|$. The limit function $\sigma(z)$ is, in a neighbourhood of $z = 0$, given by

$$\sigma(z) = f(z) + \sum_{v=0}^{\infty} \alpha(1-\alpha)^v \{s_v(z) - f(z)\},$$

where $s_v = \sum_{\mu=0}^{v} a_\mu z^\mu$ are the partial-sums of $f(z) = \sum_{v=0}^{\infty} a_v z^v$.

2. If (N, p) is a Nörlund-method with $\lim_{n\to\infty} p_n/P_n = 0$, then we obtain by Theorem 5 that each power series $f(z) = \sum_{v=0}^{\infty} a_v z^v$ with radius of convergence 1 is compactly summable (N, p) in the unit disc to the "right value" $f(z)$. (Note that in this case the method (N, p) is not regular in general.)

3. If (N, p) is a Nörlund-method with $\lim_{n\to\infty} p_n/P_n = 1$, then by Theorem 5 and Remark 1 each power series $f(z) = \sum_{v=0}^{\infty} a_v z^v$ is compactly summable (N, p) in $\mathbb{C}$ to the value a_0.

EXAMPLE. Let us consider the Nörlund-summability of the geometric series $f(z) = \sum_{v=0}^{\infty} z^v$ by (N, p)-methods defined by the sequence $\{p_n\}_{n=0}^{\infty}$ where $p_n = q^n$, $q > 0$. Then for $q \neq 1$, $z \neq 1$ the (N, p)-transforms of $f(z)$ in this case are

$$\sigma_n(z) = \sum_{v=0}^{n} \frac{p_{n-v}}{P_n} s_v(z) = \sum_{v=0}^{n} \frac{p_{n-v}}{P_n} \cdot \frac{1 - z^{v+1}}{1 - z}$$
$$= \frac{1}{1-z} - \frac{z}{1-z} \cdot \frac{q-1}{q^{n+1}-1} \cdot q^n \sum_{v=0}^{n} \left(\frac{z}{q}\right)^v.$$

(a) If $0 < q < 1$ we have

$$\lim_{n\to\infty} \frac{p_n}{P_n} = \lim_{n\to\infty} \frac{q^n(q-1)}{q^{n+1}-1} = 0;$$

since the row-norm condition $\sup_n 1/|P_n| \sum_{v=0}^{n} |p_v| < \infty$ is also satisfied, the (N, p)-method is regular and therefore $\{\sigma_n(z)\}$ converges uniformly in $\mathbb{D} := \{z : |z| < 1\}$ to $1/(1-z)$. By Léja's theorem we know that $\{\sigma_n(z)\}$ converges in at most countable many points z, $|z| > 1$. Let $z_0 \in \mathbb{C}$, $|z_0| > 1$; then because of

$$\sigma_n(z_0) = \frac{1}{1-z_0} - \frac{z_0}{1-z_0} \cdot \frac{1-q}{1-q^{n+1}} \cdot \frac{z_0^{n+1} - q^{n+1}}{z_0 - q} \to \infty \qquad (n \to \infty)$$

we do not even have convergence in any point z_0, $|z_0| > 1$ in this case.

(b) If $q = 1$ we have $(N, p) = (C, 1)$ and it is well known that in this case $\{\sigma_n(z)\}$ converges uniformly in $\mathbb{D}$ to $1/(1-z)$ and diverges for each point z_0, $|z_0| > 1$.

(c) If $q > 1$ we have

$$\lim_{n\to\infty} \frac{p_n}{P_n} = \lim_{n\to\infty} \frac{q^n(q-1)}{q^{n+1}-1} = 1 - \frac{1}{q}$$

and Theorem 5 tells us that in this case $\{\sigma_n(z)\}$ is compactly convergent in $|z| < 1/|1-(1-1/q)| = q$ and from

$$\sigma_n(z) = \frac{1}{1-z} - \frac{z}{1-z} \frac{q-1}{q^{n+1}-1} \cdot q^n \cdot \frac{1-(z/q)^{n+1}}{1-(z/q)}$$

we obtain for the limit function

$$\sigma(z) = \frac{1}{1-z} - \frac{z}{1-z} \cdot \frac{q-1}{q-z}.$$

4. The Behaviour of the (N, p)-Transforms $\{\sigma_n(z)\}$ Outside the Circle of Summation

Let be $f(z) = \sum_{v=0}^{\infty} a_v z^v$ a power series with radius of convergence 1. From Theorem 5 we know that for the sequence of its (N, p)-transforms $\{\sigma_n(z)\}$ we have

$$\sigma_n(z) = \sum_{v=0}^{n} \frac{p_{n-v}}{P_n} s_v(z) \xRightarrow[|z| < 1/|1-\alpha|]{} \sigma(z).$$

Now we are going to study the behaviour of the sequences $\{\sigma_n(z)\}$ of the (N, p)-transforms of $f(z)$ in $|z| \geqslant 1/|1-\alpha|$.

Theorem 6. *Let (N, p) be a Nörlund-method with $\lim_{n\to\infty} p_n/P_n = \alpha \neq 1$ and let $f(z) = \sum_{v=0}^{\infty} a_v z^v$ have radius of convergence 1. Then for each $R \geqslant 1/|1-\alpha|$ we have*

$$\overline{\lim_{n\to\infty}} \{\max_{|z|=R} |\sigma_n(z)|\}^{1/n} = R \cdot |1-\alpha|.$$

Proof. Let $R \geqslant 1/|1-\alpha|$ be fixed. From $\overline{\lim}_{n\to\infty} |a_n|^{1/n} = 1$ and $\lim_{n\to\infty} p_{n-v}/P_n = \alpha(1-\alpha)^v$ for all $v \in \mathbb{N}_0$ it is easily computed that

$$\overline{\lim_{n\to\infty}} \{\max_{|z|=R} |\sigma_n(z)|\}^{1/n} \leqslant R \cdot |1-\alpha|.$$

In the case that $\overline{\lim}_{n\to\infty} \{\max_{|z|=R} |\sigma_n(z)|\}^{1/n} < R \cdot |1-\alpha|$ there would exist an $\tilde{R} < R$ with

$$\max_{|z|=R} |\sigma_n(z)| \leqslant \tilde{R}^n |1-\alpha|^n$$

for all sufficiently large n and by using Cauchy's integral formula we would get

$$|a_n| \left|\frac{p_0}{P_n}\right| \leqslant |1-\alpha|^n \left(\frac{\tilde{R}}{R}\right)^n \tag{1}$$

for these n.

By Theorem 3 we have $\lim_{n\to\infty} |p_0/P_n|^{1/n} = |1-\alpha|$, and therefore from (1) it follows that

$$\overline{\lim_{n\to\infty}}\, |a_n|^{1/n} \leqslant \frac{\tilde{R}}{R} < 1$$

which is a contradiction to the assumption that $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ should have radius of convergence 1. Hence we must have

$$\overline{\lim_{n\to\infty}}\, \{\max_{|z|=R} |\sigma_n(z)|\}^{1/n} = R\cdot|1-\alpha| \qquad \text{for all} \quad R \geqslant \frac{1}{|1-\alpha|}.$$

This result tells us that the sequence $\{\sigma_n(z)\}$ cannot converge compactly in a circle $|z| \leqslant R$ where $R \geqslant |1-\alpha|^{-1}$.

In our next result we show that even on an arc of $|z| = R$ with $R > 1/|1-\alpha|$ the sequence $\{\sigma_n(z)\}$ cannot converge compactly and it also follows from our next theorem that on an arc of $|z| = 1/|1-\alpha|$ the sequence $\{\sigma_n(z)\}$ cannot tend to zero geometrically.

THEOREM 7. *Let (N, p) be a Nörlund-method with $\lim_{n\to\infty} p_n/P_n = \alpha \neq 1$ and let $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ have radius of convergence 1. If $R \geqslant 1/|1-\alpha|$ and Γ is a closed arc on $|z| = R$, then we have*

$$\overline{\lim_{n\to\infty}}\, \{\max_{\Gamma} |\sigma_n(z)|\}^{1/n} = R\cdot|1-\alpha|.$$

Proof. 1. Let Γ be a closed arc on $|z| = R = 1/|1-\alpha|$ and suppose

$$\overline{\lim_{n\to\infty}}\, \{\max_{\Gamma} |\sigma_n(z)|\}^{1/n} < 1. \tag{2}$$

If $\varepsilon > 0$ is given and $q \in (0, 1)$ we get from Theorem 6 and from (2) for all sufficiently large n

$$\max_{|z|=R} |\sigma_n(z)| \leqslant (1+\varepsilon)^n,$$

$$\max_{\Gamma} |\sigma_n(z)| \leqslant q^n.$$

Let $r \in (0, 1/|1-\alpha|)$ be given; then by Nevanlinna's N-constants theorem there exists a $\theta \in (0, 1)$ (independent of n) such that

$$\max_{|z|=r} |\sigma_n(z)| \leqslant (q^\theta(1+\varepsilon)^{1-\theta})^n \tag{3}$$

holds for all sufficiently large n. Choosing $\varepsilon > 0$ so small that $q^\theta(1+\varepsilon)^{1-\theta} < 1$ we get from (3)

(a) that $\{\sigma_n(z)\}$ is uniformly convergent to zero in $|z| < 1/|1-\alpha|$ which is impossible if $a_0 \neq 0$;

(b) if $a_0 = 0$ then there is a first coefficient $a_{\mu_0} \neq 0$, $\mu_0 \geqslant 1$, and we have

$$\left| a_{\mu_0} \sum_{\nu=\mu_0}^{n} \frac{p_{n-\nu}}{P_n} \right| = \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{\sigma_n(z)}{z^{\mu_0+1}} \, dz \right| \leqslant \frac{1}{r^{\mu_0}} \cdot (q^\theta(1+\varepsilon)^{1-\theta})^n$$

or

$$\left| 1 - \sum_{\nu=0}^{\mu_0-1} \frac{p_{n-\nu}}{P_n} \right| < \frac{1}{|a_{\mu_0}| \, r^{\mu_0}} \; (q^\theta(1+\varepsilon)^{1-\theta})^n.$$

This implies that $\sum_{\nu=0}^{\mu_0-1} p_{n-\nu}/P_n \to 1$ $(n \to \infty)$ which is impossible because of Theorem 2 and the assumption $\alpha \neq 1$.

2. Let Γ be a closed arc on $|z| = R > 1/|1-\alpha|$ and suppose

$$\overline{\lim_{n\to\infty}} \; \{\max_{\Gamma} |\sigma_n(z)|\}^{1/n} \leqslant \tilde{R} \cdot |1-\alpha|,$$

where $\tilde{R} < R$. If $\varepsilon > 0$ is given we get from Theorem 6 and part 1 of this proof for all sufficiently large n

$$\max_{|z|=1/|1-\alpha|} \left| \frac{\sigma_n(z)}{z^n} \right| \leqslant e^{\varepsilon n} \, |1-\alpha|^n,$$

$$\max_{|z|=R} \left| \frac{\sigma_n(z)}{z^n} \right| \leqslant \frac{R^n e^{\varepsilon n} \, |1-\alpha|^n}{R^n} = e^{\varepsilon n} \, |1-\alpha|^n,$$

$$\max_{\Gamma} \left| \frac{\sigma_n(z)}{z^n} \right| \leqslant \frac{\tilde{R}^n e^{\varepsilon n} \, |1-\alpha|^n}{R^n}.$$

Let $r \in (1/|1-\alpha|), R)$ be given; then by Nevanlinna's N-constants theorem there exists a $\theta \in (0, 1)$ such that we have

$$\max_{|z|=r} \left| \frac{\sigma_n(z)}{z^n} \right| \leqslant \left\{ \left(\frac{\tilde{R}}{R} \right)^\theta e^{\varepsilon\theta} \, |1-\alpha|^\theta \cdot e^{\varepsilon(1-\theta)} \cdot |1-\alpha|^{1-\theta} \right\}^n$$

for all sufficiently large n and it follows

$$\overline{\lim_{n\to\infty}} \{\max_{|z|=r} |\sigma_n(z)|\}^{1/n} \leqslant r\cdot|1-\alpha| \left(\frac{\tilde{R}}{R}\right)^{\theta} e^{\varepsilon}. \tag{4}$$

Choosing $\varepsilon>0$ so small that $(\tilde{R}/R)^{\theta} e^{\varepsilon}<1$ we get from (4) a contradiction to Theorem 6.

From Theorem 7 we can deduce that the sequence $\{\sigma_n(z)\}$ cannot converge compactly in a domain $G\supset\{z: |z|=1/|1-\alpha|\}$.

The remaining question, if it is possible that $\{\sigma_n(z)\}$ can at least converge pointwise outside the circle $|z|=1/|1-\alpha|$, is answered by the following theorem.

THEOREM 8. *Let $f(z)=\sum_{\nu=0}^{\infty} a_\nu z^\nu$ be a power series with radius of convergence* 1 *and* (N, p) *a Nörlund-method with* $\lim_{n\to\infty} p_n/P_n=\alpha\neq 1$; *then for each $\varepsilon>0$ the sequence $\{\sigma_n(z)\}$ converges in at most a finite number of points z with $|z|>(1/|1-\alpha|)+\varepsilon$.*

Proof. 1. Let $\varepsilon>0$ be given. We first suppose that the sequence $\{a_n\}$ is unbounded. From this sequence we choose a subsequence $\{a_{n_k}\}_{k=0}^{\infty}$ such that

$$|a_n|<|a_{n_k}| \quad \text{for } n<n_k \qquad \text{and} \qquad \lim_{k\to\infty} |a_{n_k}|^{1/n_k}=1.$$

The corresponding subsequence of the (N, p)-transforms of $f(z)$ is given by

$$\begin{aligned}\sigma_{n_k}(z)&=\sum_{\nu=0}^{n_k} \frac{p_{n_k-\nu}}{P_{n_k}} s_\nu(z)=\frac{1}{P_{n_k}} \sum_{\nu=0}^{n_k} P_{n_k-\nu} a_\nu z^\nu \\ &=\frac{a_{n_k} z^{n_k}}{P_{n_k}} \sum_{\nu=0}^{n_k} P_{n_k-\nu} \frac{a_\nu}{a_{n_k}} z^{\nu-n_k}=:\frac{a_{n_k} z^{n_k}}{P_{n_k}} \sum_{\nu=0}^{n_k} P_\nu b_\nu w^\nu,\end{aligned} \tag{5}$$

where $b_\nu=a_{n_k-\nu}/a_{n_k}$ and $w=1/z$. Since $\lim_{n\to\infty} |P_n|^{1/n}=1/|1-\alpha|$ and $\lim_{k\to\infty} |a_{n_k}|^{1/n_k}=1$ we obtain

$$\left|\frac{a_{n_k} z^{n_k}}{P_{n_k}}\right| \to \infty \qquad (k\to\infty) \tag{6}$$

for each $z\in\mathbb{C}$, $|z|>1/|1-\alpha|$. If $\{z_i\}_{i=1}^{\infty}$ is a sequence of points with $|z_i|\geqslant (1/|1-\alpha|)+\varepsilon$ for which $\{\sigma_{n_k}(z_i)\}$ converges, then because of (5) and (6) we obtain that the functions

$$\tilde{\sigma}_{n_k}(w):=\sum_{\nu=0}^{n_k} P_\nu b_\nu w^\nu \tag{7}$$

tend to zero as $k \to \infty$ in each point $w_i = 1/z_i$, $|w_i| \leqslant 1/(|1/(1-\alpha)| + \varepsilon)$. By assumption we have $|b_\nu| \leqslant 1$ for all $\nu \in \mathbb{N}_0$ and $\lim_{n\to\infty} |P_n|^{1/n} = 1/|1-\alpha|$; hence for all w with $|w| \leqslant 1/(|1/(1-\alpha)| + \varepsilon)$ we get the estimation

$$|\tilde{\sigma}_{n_k}(w)| \leqslant \sum_{\nu=0}^{n_k} |P_\nu|\,|b_\nu| \cdot |w|^\nu \leqslant c \cdot \sum_{\nu=0}^{n_k} \left(\frac{|1/(1-\alpha)| + \varepsilon/2}{|1/(1-\alpha)| + \varepsilon}\right)^\nu, \tag{8}$$

where $c > 0$ is a suitable constant. Consequently the functions $\tilde{\sigma}_{n_k}(w)$ $(k = 0, 1, ...)$ are uniformly bounded in $|w| \leqslant 1/(|1/(1-\alpha)| + \varepsilon)$. By the theorem of Montel we get that there exists a subsequence $\{\sigma_{n_{k_1}}(w)\}$ with

$$\tilde{\sigma}_{n_{k_1}}(w) \underset{0 \leqslant |w| \leqslant (|1/(1-\alpha)| + \varepsilon)^{-1}}{\Longrightarrow} 0.$$

But this is a contradiction to $\tilde{\sigma}_{n_{k_1}}(0) = P_0 \cdot b_0 = p_0 \neq 0$. Therefore the sequence $\{\sigma_n(z)\}$ can converge in at most a finite number of points z, $|z| > (1/|1-\alpha|) + \varepsilon$.

2. In the case of a bounded sequence $\{a_n\}$ of coefficients of $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ we choose a subsequence $\{a_{n_k}\}$ with $|a_{n_k}| > 0$ and $\lim_{n\to\infty} |a_{n_k}|^{1/n_k} = 1$ and get $|b_\nu| = |a_{n_k - \nu}/a_{n_k}| \leqslant M$, where $M > 0$ is a suitable constant. Then the proof is analogous to part 1.

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